PROB023: MAGIC HEXAGON

1. Problem statement

A magic hexagon consists of the numbers 1 . . . 19 arranged in a hexagonal pattern:

```
    A,B,C
   D,E,F,G
  H,I,J,K,L
   M,N,O,P
    Q,R,S
```

We have a constraint that all diagonals sum to 38. That is, `A+B+C = D+E+F+G = ... = Q+R+S = 38, A+D+H = B+E+I+M = ... = L+P+S = 38, C+G+L = B+F+K+P = ... = H+M+Q = 38`. The problem can be generalized to other sizes. This is the diameter 5 problem.

2. Problem model

Matrix notation proves useful for representing the problem in a more general format. The above problem of dimension $n = 5$ can be represented as

$$\begin{array}{ccccccccc} & & a_{11} & & a_{12} & & a_{13} & & \\ & a_{21} & & a_{22} & & a_{23} & & a_{24} & \\ a_{31} & & a_{32} & & a_{33} & & a_{34} & & a_{35} \\ & a_{41} & & a_{42} & & a_{43} & & a_{44} & \\ & & a_{51} & & a_{52} & & a_{53} & & \end{array}$$

In the following, we present the generic specification of the problem, while examples will refer to the diameter 5 problem. Let n be the dimension of the hexagon and s be the sum value common to all diagonals. Elements of the hexagon can be referred to via row and column indices, using the a_{ij} notation. However, rows do have different lengths. Furthermore, there is a symmetry of row lengths with regard to the middle row, which is identified by the number $\lceil \frac{n}{2} \rceil$. Each row $i \in 1 \ldots n$ thus has $n - \left| \lceil \frac{n}{2} \rceil - i \right|$ elements. Assuming that the minimal number of row elements in any hexagon is 2, domains of each element a_{ij} are $1 \ldots \lceil \frac{s}{2} \rceil$. The diagonal sums can be described for $i \in 1 \ldots n$ by the following three functions.

2.1. Row sums:

$$row\_sum(i) = \sum_{j=1}^{n - \left| \lceil \frac{n}{2} \rceil - i \right|} a_{ij} \tag{1}$$

2.2. Right-sloping diagonal sums:

$$rs\_dsum(i) = \sum_{j=1}^{n - \left| \lceil \frac{n}{2} \rceil - i \right|} a_{f(i,j),\, g(i,j)} \tag{2}$$

where

$$\begin{array}{rcl} f(i,j) & = & j + \max(0, \lceil \frac{n}{2} \rceil - i) \\ g(i,j) & = & i - \max(0, \lceil \frac{n}{2} \rceil - f(i,j)) \end{array}$$

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2.3. **Left-sloping diagonal sums:**

$$ls\_dsum(i) = \sum_{j=1}^{n-|\lceil \frac{n}{2} \rceil - i|} a_{h(i,j),\, k(i,j)} \tag{3}$$

where

$$\begin{array}{rcl} h(i,j) & = & j + \max(0, i - \lceil \frac{n}{2} \rceil) \\ k(i,j) & = & i - \max(0, h(i,j) - \lceil \frac{n}{2} \rceil) \end{array}$$

The formulations (2) and (3) are similar in that h is symmetrical to f and k is symmetrical to g.

3. Z specification

The description is split up into several blocks to allow a clearer presentation. Data types and their invariants constitute the first block. We introduce the magic hexagon to be of type *varmat*,[1] i.e. a sequence of rows with different lengths. An example for this data type model is for instance the representation for a magic hexagon of diameter $n = 3$:

$$\begin{array}{llll} mh = \{ & 1 & \mapsto & \{1 \mapsto a_{11}, 2 \mapsto a_{12}\}, \\ & 2 & \mapsto & \{1 \mapsto a_{21}, 2 \mapsto a_{22}, 3 \mapsto a_{23}\}, \\ & 3 & \mapsto & \{1 \mapsto a_{31}, 2 \mapsto a_{32}\} \quad \} \end{array}$$

We parameterise this type by n for the diameter and s for the constant diagonal sums, the invariants are as discussed in section 2.

MH_Data

$$\begin{array}{l} mh : varmat\ \mathbb{N} \\ n, s : \mathbb{N} \\ \hline rdim(mh) = n \\ \forall\, i : 1 \ldots n \bullet cdim(mh, i) = n - |\lceil \frac{n}{2} - i \rceil| \\ \forall\, i : 1 \ldots n;\ j : \mathbb{N} \mid j \in 1 \ldots cdim(mh, i) \bullet mh(i,j) \in 1 \ldots \lceil \frac{s}{2} \rceil \end{array}$$

The first two constraints define the dimensions of mh, the last one is a domain definition for all its elements. The second block now contains the auxiliary functions of section 2.

MH_Aux

$$\begin{array}{l} MH\_Data \\ f, g, h, k : (\mathbb{N}_1 \times \mathbb{N}_1) \longrightarrow \mathbb{N} \\ rsum, rs\_dsum, ls\_dsum : \mathbb{N}_1 \longrightarrow \mathbb{N} \\ \hline \forall\, i : 1 \ldots n;\ j : \mathbb{N} \mid j \in 1 \ldots cdim(mh, i) \bullet \\ \qquad f(i,j) = j + \max\{0, \lceil \frac{n}{2} \rceil - i\} \ \wedge\ g(i,j) = i - \max\{0, \lceil \frac{n}{2} \rceil - f(i,j)\} \ \wedge \\ \qquad h(i,j) = j + \max\{0, i - \lceil \frac{n}{2} \rceil\} \ \wedge\ k(i,j) = i - \max\{0, h(i,j) - \lceil \frac{n}{2} \rceil\} \\ \forall\, i : 1 \ldots n \bullet \\ \quad rsum(i) = \Sigma(\ items(mh(i))\ ) \wedge \\ \quad rs\_dsum(i) = \Sigma(\ items\{\ j : \mathbb{N} \mid j \in 1 \ldots cdim(mh, i) \bullet j \mapsto mh(f(i,j), g(i,j))\ \}\ ) \wedge \\ \quad ls\_dsum(i) = \Sigma(\ items\{\ j : \mathbb{N} \mid j \in 1 \ldots cdim(mh, i) \bullet j \mapsto mh(h(i,j), k(i,j))\ \}\ ) \end{array}$$

The first part of the above is a syntactic transformation of the index functions developed in section 2 into Z, the second part realizes the three functions (1)...(3) of section 2. All three follow the same construction principle; elements of mh are mapped into a sequence whose elements are collected into a bag using the

[1] this type, as well as the operations *rdim*, *cdim* to describe the row and column dimensions, respectively, is introduced separately.

items function [1, p. 127], whose elements in turn are summed up. We can now assemble the main problem definition.

$$
\begin{array}{l}
\textit{Magic\_Hexagon} \\
\hline
\textit{MH\_Data} \\
\textit{MH\_Aux} \\
\hline
\forall\, i, j : 1 \mathinner{\ldotp\ldotp} n \bullet \\
\quad rsum(i) = s = rsum(j) \;\land\; rs\_dsum(i) = s = rs\_dsum(j) \;\land\; ls\_dsum(i) = s = ls\_dsum(j)
\end{array}
$$

This statement closely corresponds to the informal problem description in natural language, the main work that this problem causes lies in setting up the numbering scheme for the hexagon.

References

[1] J. M. Spivey. *The Z Notation: A Reference Manual*. J. M. Spivey, Oriel College, Oxford OX1 4EW, second edition, 1998. First published 1992 by Prentice Hall, current version on http://spivey.oriel.ox.ac.uk/mike/zrm/.